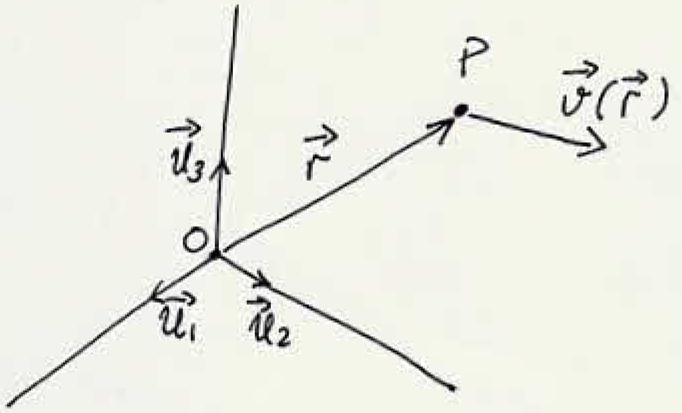
VECTORES, COORDENADAS Y FUNCIONES



$\vec{r}$ ≡ VECTOR DE POSICION

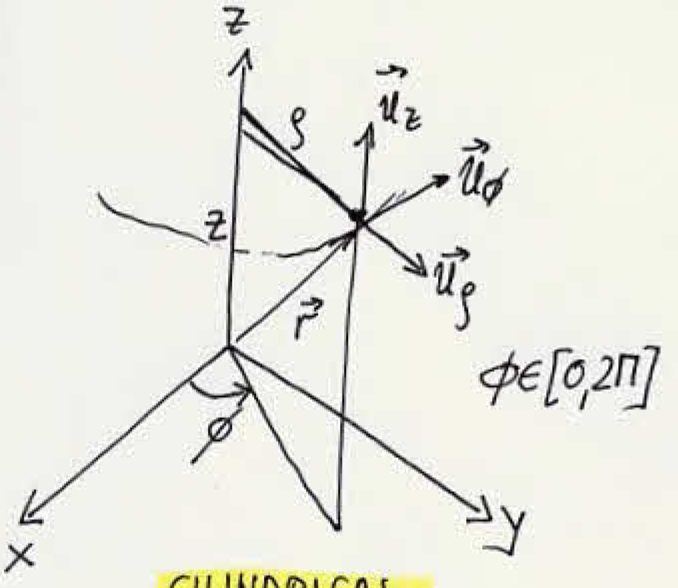
P ≡ PUNTO DEL ESPACIO

$T(\vec{r})$ ≡ FUNCION ESCALAR

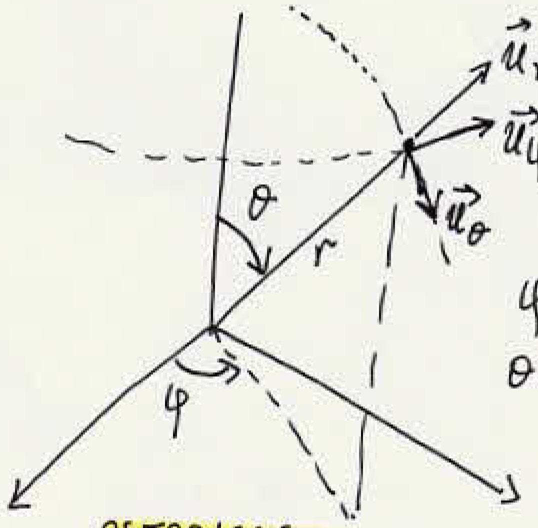
$\vec{v}(\vec{r})$ ≡ FUNCION VECTORIAL

SISTEMAS DE COORDENADAS

- CARTESIANAS
- CILINDRICAS
- ESFERICAS



$\phi \in [0, 2\pi]$



$\varphi \in [0, 2\pi]$

$\theta = [0, \pi]$

CILINDRICAS

$$\begin{cases} x = \rho \cos\phi \\ y = \rho \operatorname{sen}\phi \\ z = z \end{cases}$$

$$d\vec{r} = d\rho\, \vec{u}_\rho + \rho\, d\phi\, \vec{u}_\phi + dz\, \vec{u}_z$$

$$dV = \rho\, d\phi\, d\rho\, dz$$

ESFERICAS

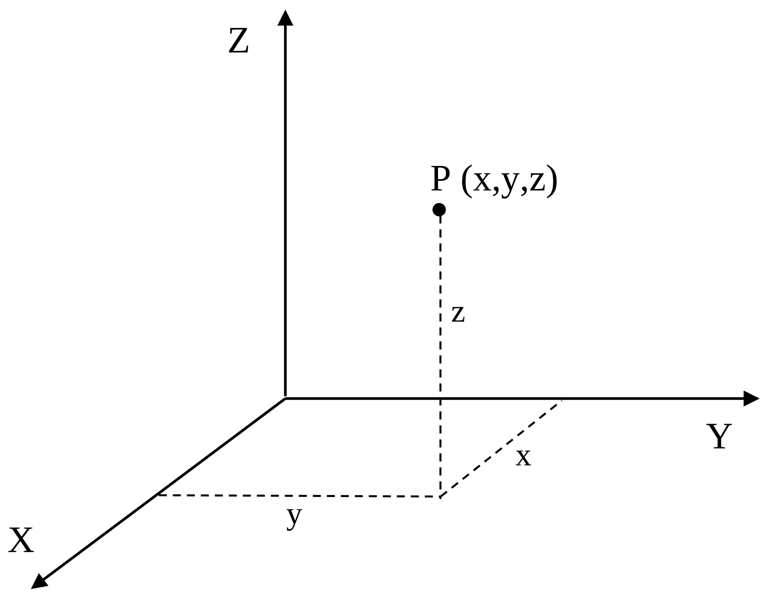
$$\begin{cases} x = r \operatorname{sen}\theta \cos\varphi \\ y = r \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = r\cos\theta \end{cases}$$

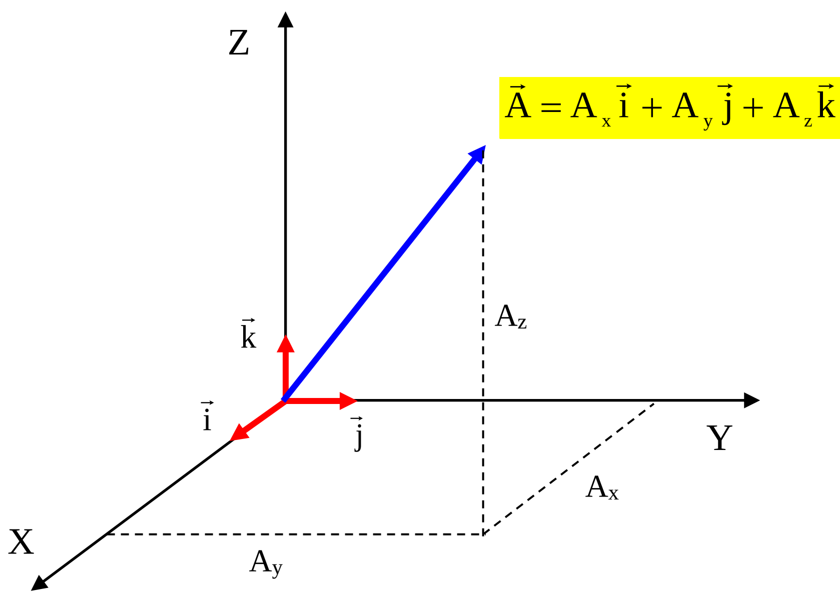
$$d\vec{r} = dr\, \vec{u}_r + r\, d\theta\, \vec{u}_\theta + r \operatorname{sen}\theta\, d\varphi\, \vec{u}_\varphi$$

$$dV = r^2 \operatorname{sen}\theta\, d\theta\, d\varphi\, dr$$

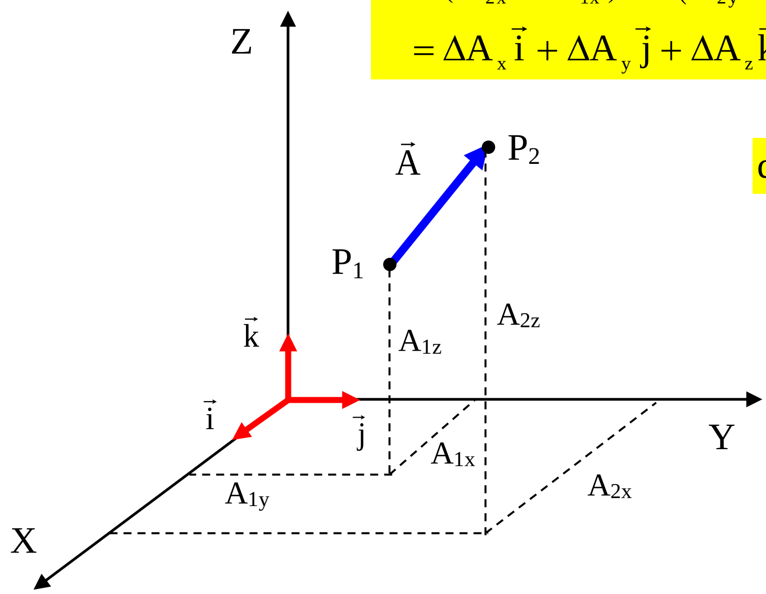
SISTEMAS DE COORDENADAS.

SISTEMA DE COORDENADAS CARTESIANAS





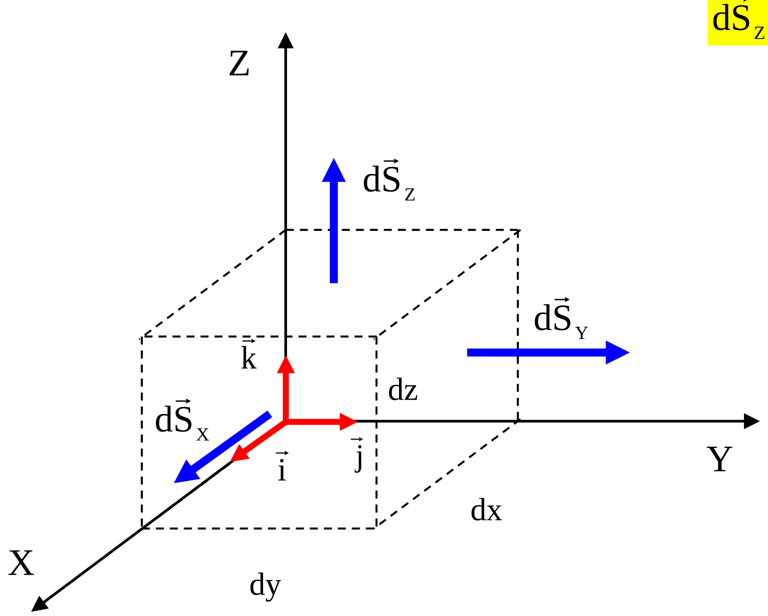
$$\vec{A} = (A_{2x} - A_{1x})\vec{i} + (A_{2y} - A_{1y})\vec{j} + (A_{2z} - A_{1z})\vec{k} =$$
$$= \Delta A_x \vec{i} + \Delta A_y \vec{j} + \Delta A_z \vec{k}$$



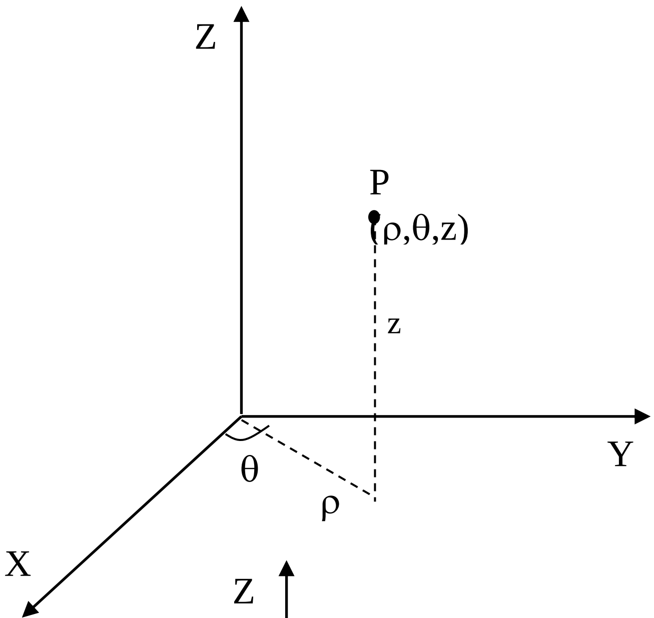
$$d\vec{l} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

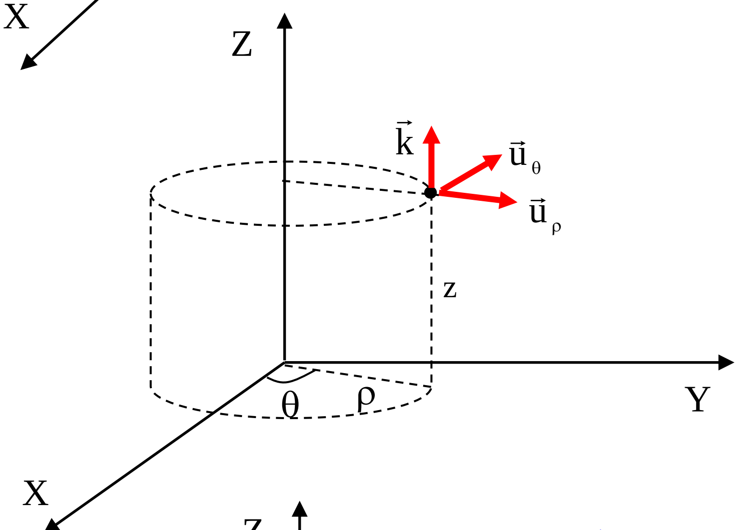
$$dV = dxdydz$$

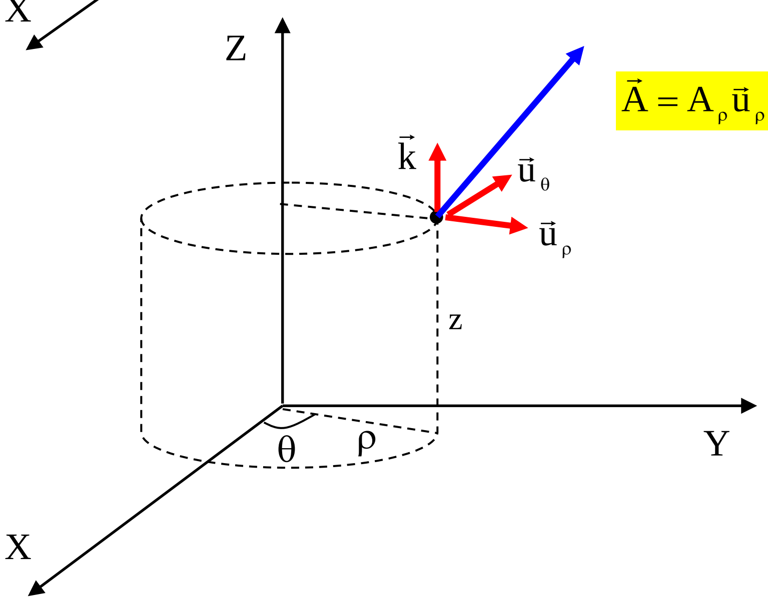
$$d\vec{S}_X = dydz\,\vec{i}$$
$$d\vec{S}_Y = dxdz\,\vec{j}$$
$$d\vec{S}_Z = dxdy\,\vec{k}$$



SISTEMA DE COORDENADAS CILÍNDRICAS





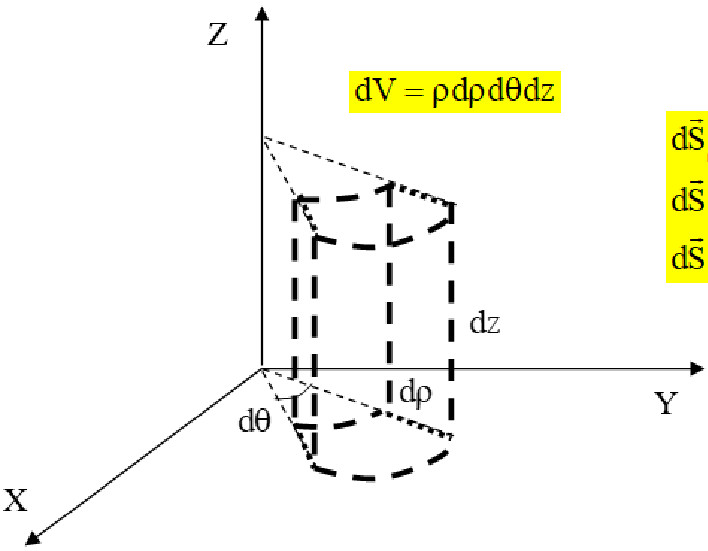


$$\vec{A} = A_\rho \vec{u}_\rho + A_\theta \vec{u}_\theta + A_k \vec{k}$$

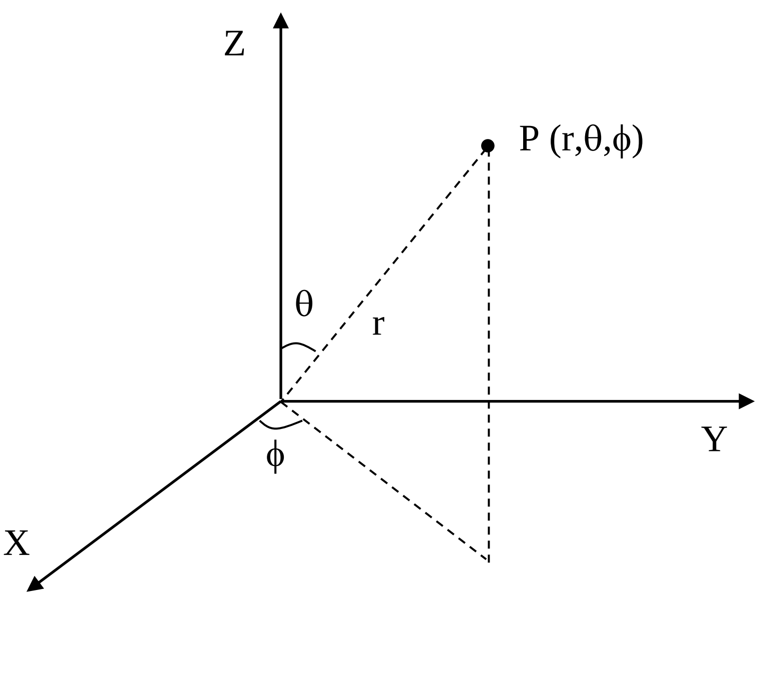
$$d\vec{r} = d\rho\,\vec{u}_\rho + \rho d\theta\,\vec{u}_\theta + dz\,\vec{k}$$

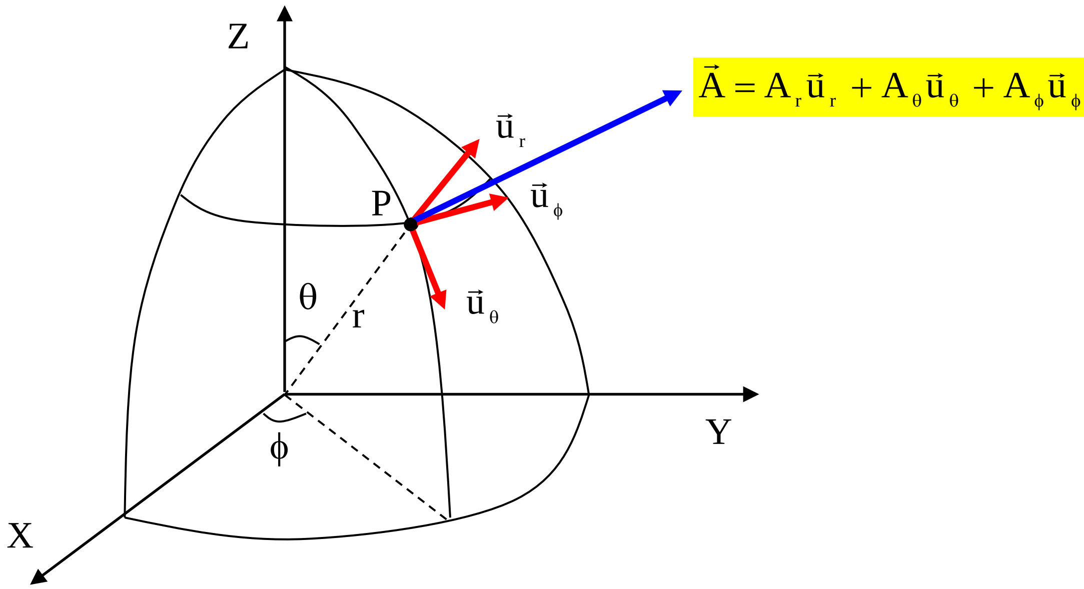
$$dV = \rho d\rho d\theta dz$$

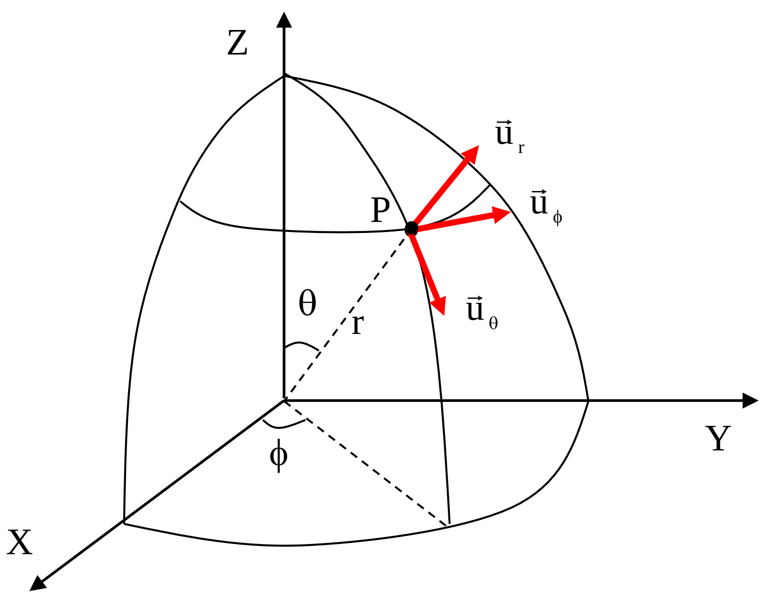
$$d\vec{S}_\rho = \rho d\theta dz\,\vec{u}_\rho$$
$$d\vec{S}_\theta = d\rho dz\,\vec{u}_\theta$$
$$d\vec{S}_z = \rho d\rho d\theta\,\vec{k}$$

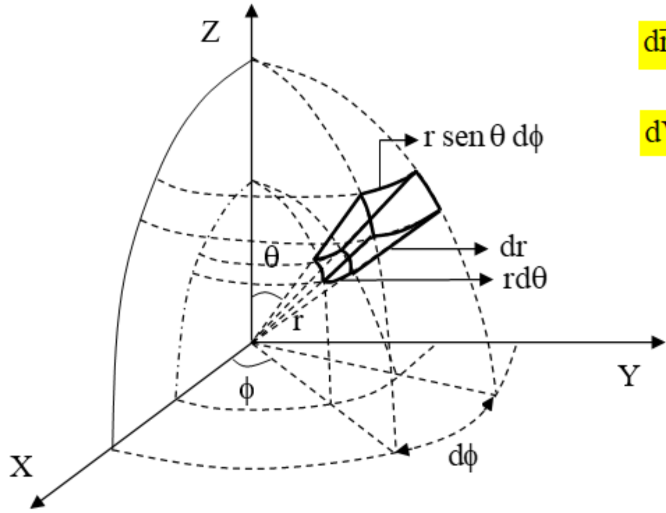


SISTEMA DE COORDENADAS ESFÉRICAS.




Z
$\vec{A} = A_r\vec{u}_r + A_\theta\vec{u}_\theta + A_\phi\vec{u}_\phi$
$\vec{u}_r$
P
$\vec{u}_\phi$
θ
r
$\vec{u}_\theta$
Y
ϕ
X


Z
$\vec{u}_r$
P
$\vec{u}_\phi$
θ
r
$\vec{u}_\theta$
Y
ϕ
X



$$d\vec{r} = dr\,\vec{u}_r + rd\theta\vec{u}_\theta + r\,\text{sen}\,\theta d\phi\vec{u}_\phi$$

$$dV = r^2\,\text{sen}\,\theta\,drd\theta d\phi$$

$$d\vec{S}_r = r^2\,\text{sen}\,\theta\,d\theta d\phi\,\vec{u}_r$$
$$d\vec{S}_\theta = r\,\text{sen}\,\theta\,drd\phi\,\vec{u}_\theta$$
$$d\vec{S}_\phi = r\,drd\theta\,\vec{u}_\phi$$

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

RESUMEN DE ANALISIS VECTORIAL BÁSICO

1 - CAMPO ESCALAR

$T(\vec{r},t)$ → CAMPO ESCALAR VARIABLE CON t

$T(\vec{r})$ → CAMPO ESCALAR ESTACIONARIO

VIENE GRAFICAMENTE REPRESENTADO POR LAS SUP. EQUIESCALARES

$$T_i = \{\vec{r} \in \mathbb{R}^3 \mid T(\vec{r}) = T_i\}$$ (ISOTERMAS, SUP. EQUIPOT., - - -)

Ej: TEMPERATURA, DENS. DE MASA, ...

a) GRADIENTE : $\vec{\nabla} \cdot T(\vec{r})$

CARTES. $$\vec{\nabla}T(\vec{r}) = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}$$

CILIND. $$\vec{\nabla}T(\vec{r}) = \frac{\partial T}{\partial \rho}\vec{u}_\rho + \frac{1}{\rho}\frac{\partial T}{\partial \phi}\vec{u}_\phi + \frac{\partial T}{\partial z}\vec{u}_z$$

ESFER. $$\vec{\nabla}T(\vec{r}) = \frac{\partial T}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\vec{u}_\theta + \frac{1}{r\,\mathrm{sen}\,\theta}\frac{\partial T}{\partial \varphi}\vec{u}_\varphi$$

$\vec{\nabla}T$:
- MODULO : $|\vec{\nabla}T| = \frac{dT}{|d\vec{r}|}$ VARIACION DE T POR U. DE L. EN LA DIRECCION DE MAXIMA VARIACION
- DIRECCION : $\vec{\nabla}T \perp$ SUP. EQUIESCALARES
- SENTIDO : CRECIMIENTO DEL ESCALAR T

$$\boxed{dT = \vec{\nabla}T \cdot d\vec{r}}$$

(DERIVADA DIRECCIONAL $\vec{\nabla}T \cdot \vec{n} = \frac{dT}{ds}$)

EJERCICIO: DADO $T = 2xy^2 + y \cdot z$, y $\vec{v} = \vec{i} + \vec{j} + \vec{k}$, CALCULAR $\frac{dT}{ds}$ CON S, PARAMETRO A LO LARGO DE $\vec{v}$

b) INTEGRALES DE CAMPOS ESCALARES (O DE FUNCIONES ESCAL.)

$$\int_V \rho(\vec{r}) . dV \quad \Big| \quad \rho(\vec{r}) \text{ ES UNA DENSIDAD}$$

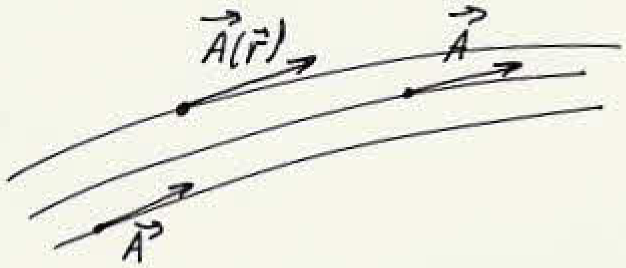
- MASA
- CARGA
- ENERGIA

2) CAMPOS VECTORIALES

$\vec{A}(\vec{r},t) \equiv$ FUNCION VECTORIAL DE PUNTO Y VARIABLE CON t

$\vec{A}(\vec{r}) \equiv$ F.V.P. ESTACIONARIA

VIENE GRAFICAMENTE REPRESENTADO POR LINEAS DE CORRIENTE



Ej:

CORRIENTE DE MASA DE UN FLUIDO	$\rho . \vec{v}$	$[kg/s.m^2]$
DENSIDAD DE CORRIENTE ELECTRICA	$\vec{J}$	$[A/m^2]$
DENSIDAD DE CORRIENTE DE CALOR	$\vec{q}$	$[W/m^2]$

a) CALCULO DIFERENCIAL

i) DIVERGENCIA $\quad \vec{\nabla} \cdot \vec{A}$ (ES UN ESCALAR)

ii) ROTACIONAL $\quad \vec{\nabla} \times \vec{A}$ (ES UN VECTOR)

Coordenadas	Expresión de la divergencia
Cartesianas	$div\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \nabla \cdot \vec{A}$
Esféricas	$div\vec{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\,\text{sen}\theta}\frac{\partial (A_\theta\,\text{sen}\theta)}{\partial \theta} + \frac{1}{r\,\text{sen}\theta}\frac{\partial A_\varphi}{\partial \varphi} = \nabla \cdot \vec{A}$
Cilíndricas	$div\vec{A} = \frac{1}{\rho}\frac{\partial (\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \nabla \cdot \vec{A}$

Coordenadas	Expresión del rotacional
Cartesianas	$rot\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$
Esféricas	$rot\vec{A} = \vec{\nabla} \times \vec{A} = \frac{1}{r^2 \operatorname{sen}\theta} \begin{vmatrix} \vec{u}_r & \vec{u}_\theta r & \vec{u}_\varphi r \operatorname{sen}\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & (r \operatorname{sen}\theta) A_\varphi \end{vmatrix}$
Cilíndricas	$rot\vec{A} = \vec{\nabla} \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \vec{u}_\rho & \vec{u}_\phi \rho & \vec{u}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$

IDENTIDADES DEL CÁLCULO VECTORIAL

$$\vec{A} \bullet \vec{B} \times \vec{C} = \vec{B} \bullet \vec{C} \times \vec{A} = \vec{C} \bullet \vec{A} \times \vec{B}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A} \bullet \vec{C}) - \vec{C}\,(\vec{A} \bullet \vec{B})$$

$$\vec{\nabla}(\Psi V) = \Psi\,\vec{\nabla} V + V\,\vec{\nabla}\Psi$$

$$\vec{\nabla} \bullet (\Psi \vec{A}) = \Psi\,\vec{\nabla} \bullet \vec{A} + \vec{A} \bullet \vec{\nabla}\Psi$$

$$\vec{\nabla} \times (\Psi \vec{A}) = \Psi\,\vec{\nabla} \times \vec{A} + \vec{\nabla}\Psi \times \vec{A}$$

$$\vec{\nabla} \bullet (\vec{A} \times \vec{B}) = \vec{B} \bullet (\vec{\nabla} \times \vec{A}) - \vec{A} \bullet (\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla} \bullet \vec{\nabla} V = \vec{\nabla}^2 V$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \bullet \vec{A}) - \vec{\nabla}^2 \vec{A}$$

$$\vec{\nabla} \times \vec{\nabla} V = 0$$

$$\vec{\nabla} \bullet (\vec{\nabla} \times \vec{A}) = 0$$

ϕ ESCALAR $\phi(u_1, u_2, u_3)$
$\vec{A} = A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3$

COORDENADAS CURVILÍNEAS

a) $$\vec{\nabla}\phi = \frac{1}{h_1}\frac{\partial \phi}{\partial u_1}\vec{e}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial u_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial \phi}{\partial u_3}\vec{e}_3$$

b) $$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

c) $$\vec{\nabla}\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \partial_{u_1} & \partial_{u_2} & \partial_{u_3} \\ h_1A_1 & h_2A_2 & A_3A_3 \end{vmatrix}$$

d) $$\vec{\nabla}^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial \phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial \phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial \phi}{\partial u_3}\right)\right]$$

SISTEMAS COORDENADOS MAS COMUNES.

1) CARTESIANAS: $u_1 = x,\ u_2 = y,\ u_3 = z \qquad h_1 = h_2 = h_3 = 1$

2) CILINDRICAS $u_1 = \rho \quad u_2 = \theta,\ u_3 = z \qquad h_1 = 1,\ h_2 = \rho,\ h_3 = 1$

3) ESFERICAS $u_1 = r \quad u_2 = \theta,\ u_3 = \varphi \qquad h_1 = 1,\ h_2 = r,\ h_3 = r\,\text{sen}\,\theta$

Ej: CILINDRICAS

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial \rho}\vec{u}_\rho + \frac{1}{\rho}\frac{\partial \phi}{\partial \theta}\vec{u}_\theta + \frac{\partial \phi}{\partial z}\vec{u}_z$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{\partial}{\partial \theta}(A_\theta) + \frac{\partial}{\partial z}(\rho A_z)\right] =$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}^2\phi = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \phi}{\partial \rho}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\rho}\frac{\partial \phi}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial \phi}{\partial z}\right)\right]$$

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \quad \nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\textit{Divergence}: \quad \nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\textit{Curl}: \quad \nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \quad \nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Identities

1. $(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$
2. $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$
3. $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f$
4. $\boldsymbol{\nabla}(a/b) = (1/b)\boldsymbol{\nabla}a - (a/b^2)\boldsymbol{\nabla}b$
5. $\boldsymbol{\nabla}(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} + (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) + \boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B})$
6. $\boldsymbol{\nabla} \cdot (f\boldsymbol{A}) = (\boldsymbol{\nabla}f) \cdot \boldsymbol{A} + f(\boldsymbol{\nabla} \cdot \boldsymbol{A})$
7. $\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B})$
8. $(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})f = \boldsymbol{\nabla}^2 f$
9. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}f) = 0$
10. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$
11. $\boldsymbol{\nabla} \times (f\boldsymbol{A}) = (\boldsymbol{\nabla}f) \times \boldsymbol{A} + f(\boldsymbol{\nabla} \times \boldsymbol{A})$
12. $\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{\nabla} \cdot \boldsymbol{B})\boldsymbol{A} - (\boldsymbol{\nabla} \cdot \boldsymbol{A})\boldsymbol{B}.$
13. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2\boldsymbol{A}$ (Sec. 1.11.6)
14. $$(\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} = \left[A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z}\right]\hat{\boldsymbol{x}} + \left[A_x \frac{\partial B_y}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_y}{\partial z}\right]\hat{\boldsymbol{y}} + \left[A_x \frac{\partial B_z}{\partial x} + A_y \frac{\partial B_z}{\partial y} + A_z \frac{\partial B_z}{\partial z}\right]\hat{\boldsymbol{z}}$$
15. $\boldsymbol{\nabla}'(1/r) = \hat{\boldsymbol{r}}/r^2$. This is the gradient calculated at (x', y', z'), and $\boldsymbol{r}$ is the vector $\boldsymbol{r}$ pointing from (x', y', z') to (x, y, z).
16. $\boldsymbol{\nabla}(1/r) = -\hat{\boldsymbol{r}}/r^2$. This is the gradient calculated at (x, y, z) with the same vector $\boldsymbol{r}$.
17. $\boldsymbol{\mathcal{A}} = \frac{1}{2}\oint_C \boldsymbol{r} \times \boldsymbol{dl}$, where the surface of area $\mathcal{A}$ is plane. The vector $\boldsymbol{r}$ extends from an arbitrary origin to a point on the curve C that bounds $\mathcal{A}$.
18. $\int_v \boldsymbol{\nabla}f\, dv = \int_{\mathcal{A}} f\, \boldsymbol{d\mathcal{A}}$
19. $\int_v (\boldsymbol{\nabla} \times \boldsymbol{A})\, dv = -\int_{\mathcal{A}} \boldsymbol{A} \times \boldsymbol{d\mathcal{A}}$, where $\mathcal{A}$ is the area of the closed surface that bounds the volume v.
20. $\oint_C f\, \boldsymbol{dl} = -\int_{\mathcal{A}} \boldsymbol{\nabla}f \times \boldsymbol{d\mathcal{A}}$ where C is the closed curve that bounds the open surface of area $\mathcal{A}$.

Theorems

1. The divergence theorem. $\int_{\mathcal{A}} \boldsymbol{A} \cdot \boldsymbol{d\mathcal{A}} = \int_v \boldsymbol{\nabla} \cdot \boldsymbol{A}\, dv$ where $\mathcal{A}$ is the area of the closed surface that bounds the volume v.
2. Stokes's theorem: $\oint_C \boldsymbol{A} \cdot \boldsymbol{dl} = \int_{\mathcal{A}} (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \boldsymbol{d\mathcal{A}}$.

MAS PROPIEDADES DE LA DIVERGENCIA Y EL ROTACIONAL

If the curl of a vector field (**F**) vanishes (everywhere), then **F** can be written as the gradient of a **scalar potential** (V):

$$\nabla \times \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = -\nabla V. \tag{1.103}$$

(The minus sign is purely conventional.) That's the essential burden of the following theorem:

Theorem 1: **Curl-less** (or "**irrotational**") **fields**. The following conditions are equivalent (that is, **F** satisfies one if and only if it satisfies all the others):
(a) $\nabla \times \mathbf{F} = 0$ everywhere.
(b) $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$ is independent of path, for any given end points.
(c) $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed loop.
(d) **F** is the gradient of some scalar, $\mathbf{F} = -\nabla V$.

The scalar potential is not unique—any constant can be added to V with impunity, since this will not affect its gradient.

If the divergence of a vector field (**F**) vanishes (everywhere), then **F** can be expressed as the curl of a **vector potential** (**A**):

$$\nabla \cdot \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = \nabla \times \mathbf{A}. \tag{1.104}$$

That's the main conclusion of the following theorem:

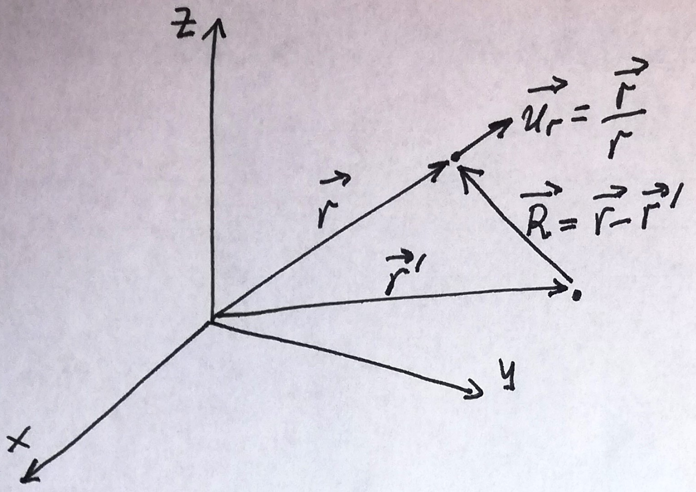
Theorem 2: **Divergence-less** (or "**solenoidal**") **fields**. The following conditions are equivalent:
(a) $\nabla \cdot \mathbf{F} = 0$ everywhere.
(b) $\int \mathbf{F} \cdot d\mathbf{a}$ is independent of surface, for any given boundary line.
(c) $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for any closed surface.
(d) **F** is the curl of some vector, $\mathbf{F} = \nabla \times \mathbf{A}$.

The vector potential is not unique—the gradient of any scalar function can be added to **A** without affecting the curl, since the curl of a gradient is zero.

You should by now be able to prove all the connections in these theorems, save for the ones that say (a), (b), or (c) implies (d). Those are more subtle, and will come later. Incidentally, in *all* cases (*whatever* its curl and divergence may be) a vector field **F** can be written as the gradient of a scalar plus the curl of a vector:

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A} \qquad \text{(always)}. \tag{1.105}$$

OTRAS EXPRESIONES UTILES



$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{u}_r}{r^2} ,$$

$$\vec{\nabla}^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\vec{r})$$

$$\vec{\nabla}\left(\frac{\vec{u}_r}{r}\right) = \frac{1}{r^2} \quad ; \quad \vec{\nabla}(\vec{r}) = 3$$

$$\vec{\nabla}\left(\frac{\vec{u}_r}{r^2}\right) = 4\pi\,\delta(\vec{r})$$

Si $\vec{R} = \vec{r} - \vec{r}'$, $\vec{u}_R = \frac{\vec{R}}{R}$ $\quad |\vec{R}| = R$

$$\vec{\nabla}(R) = \vec{u}_R$$

$$\vec{\nabla}(R^n) = n R^{n-1} \cdot \vec{u}_R$$

$$\vec{\nabla} f(\vec{R}) = -\vec{\nabla}' f(\vec{R}) \quad \left(\vec{\nabla}' \equiv \frac{\partial}{\partial r'}\right)$$

$$\vec{\nabla}\vec{A}(\vec{R}) = -\vec{\nabla}'\vec{A}(\vec{R})$$

$$\vec{\nabla}\times\vec{A}(\vec{R}) = -\vec{\nabla}'\times\vec{A}(\vec{R})$$

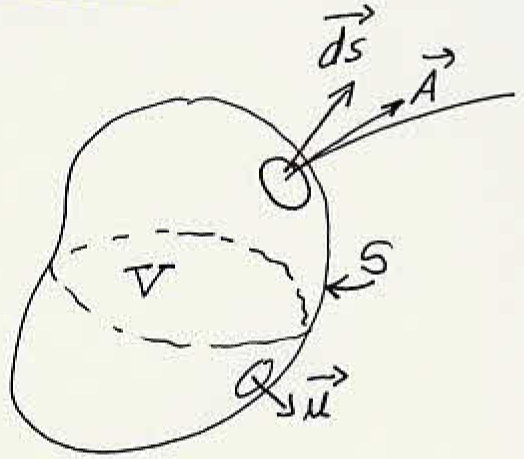
$$\vec{\nabla}^2 f(R) = \vec{\nabla}'^2 f(R)$$

b) CALCULO INTEGRAL PARA CAMPOS VECTORIALES

i) FLUJO DE UN CAMPO VECTORIAL

$$\phi = \int_S \vec{A} \cdot \vec{dS} \quad ó \quad \phi = \oint_S \vec{A} \cdot \vec{dS}$$

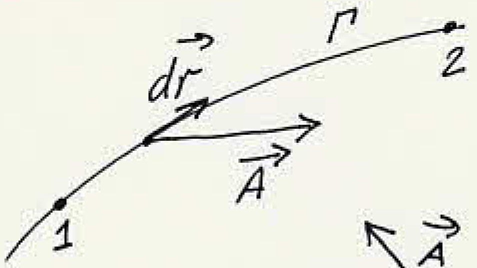
S → ABIERTA; S → CERRADA



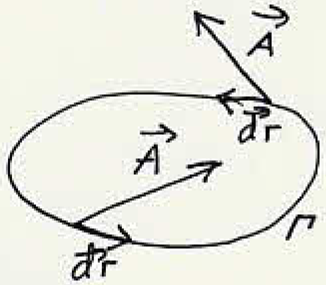
Ej: $\vec{A} = \rho \cdot \vec{v}$ " $\int_S \rho \vec{v} \cdot \vec{dS} = G_m$ (CAUDAL MASICO)

ii) CIRCULACIÓN DE UN CAMPO VECTORIAL

$$C = \int_{\Gamma_{1,2}} \vec{A} \cdot \vec{dr} = \int_1^2 \vec{A} \cdot \vec{dr}$$



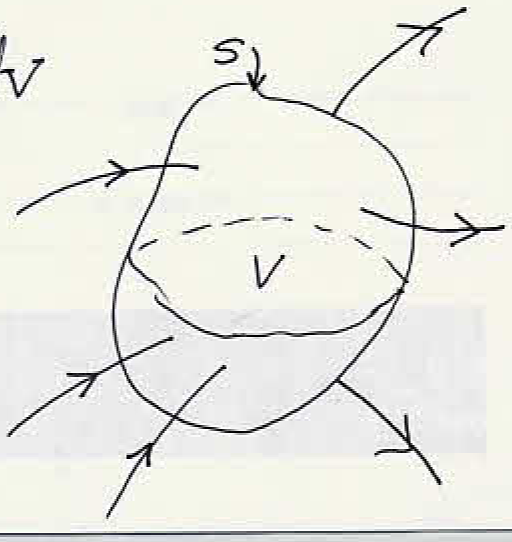
$$C_\phi = \oint_\Gamma \vec{A} \cdot \vec{dr}$$



c) TEOREMAS INTEGRALES MUY USADOS

i) TEOREMA DE LA DIVERGENCIA (TEOREMA DE GAUSS)

$$\oint_S \vec{A} \cdot \vec{dS} = \int_V \vec{\nabla} \cdot \vec{A} \cdot dV$$



ii) TEOREMA DEL ROTACIONAL (TEOREMA DE STOKES)

$$\oint_{\Gamma} \vec{A} \cdot d\vec{r} = \int_{S} \vec{\nabla} \times \vec{A} \cdot d\vec{s}$$

iii) IDENTIDAD DE GREEN.

SEAN ϕ y ψ FUNCIONES ESCALARES DE CLASE C^2 EN V LIMITADO POR S y $\vec{u}$ NORMAL EXTERIOR

$$\int_{V} \left[\phi \vec{\nabla}^2 \psi - \psi \vec{\nabla}^2 \phi\right] dV = \int_{S} \left[\phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi\right] d\vec{s}$$

APLICACIONES

- CONVERTIR EC. DIF. PARC. $\longrightarrow$ ECUACION INTEGRAL
- FUNDAMENTO DEL METODO DE LOS ELEMENTOS DE CONTORNO