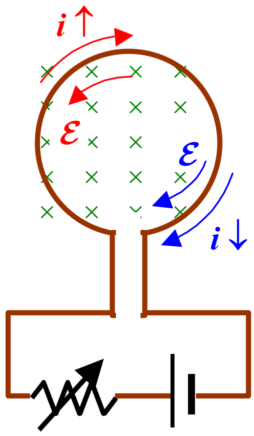
Tema 41.- Inductancia

§41.1.- Autoinducción



Autoinducción en una espira

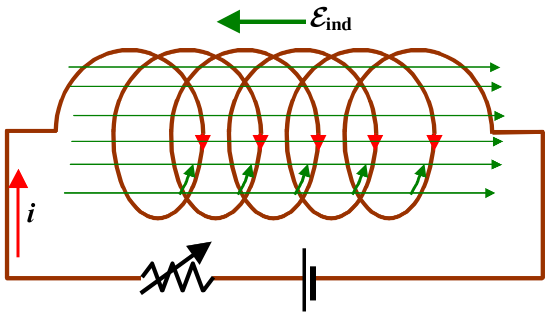
Flujo ligado: $(N\Phi_B) = Li$

L es la autoinducción o inductancia

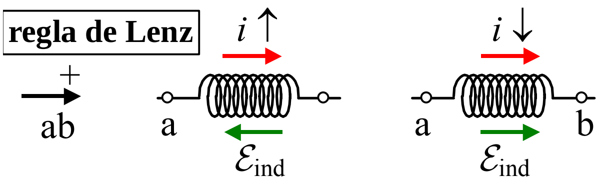
$\therefore \quad L = \dfrac{N\Phi_B}{i} \qquad \dfrac{\text{Wb}\cdot\text{vuelta}}{\text{A}} = \text{H} \quad$ (henrio)

- L depende del tamaño, forma y número de espiras
- L depende de las propiedades magnéticas del medio
  - En ausencia de materiales ferromagnéticos: L = cte.
  - En presencia de materiales ferromagnéticos: $B = B(i) \rightarrow \Phi = \Phi(i) \rightarrow L = L(i)$

$$\begin{cases} N\dfrac{\mathrm{d}\Phi_B}{\mathrm{d}t} = L\dfrac{\mathrm{d}i}{\mathrm{d}t} \\ \mathcal{E}_{\text{ind}} = -N\dfrac{\mathrm{d}\Phi_B}{\mathrm{d}t} \end{cases} \rightarrow \begin{cases} \mathcal{E}_{\text{ind}} = -L\dfrac{\mathrm{d}i}{\mathrm{d}t} \\ \boxed{L = -\dfrac{\mathcal{E}_{\text{ind}}}{\mathrm{d}i/\mathrm{d}t}} \end{cases}$$

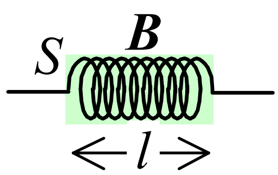


Definición del henrio (H)



§41.2.- Cálculo de la autoinductancia

41.2.a. Solenoide y toroide

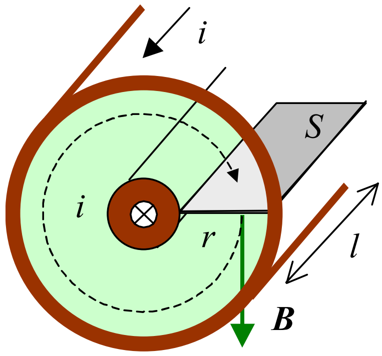


Teorema de Ampère: $Bl = \mu_0 Ni \quad \rightarrow \quad B = \mu_0 \dfrac{N}{l} i$

Flujo ligado: $\Phi = BS = \mu_0 \dfrac{N}{l} iS \quad \rightarrow \quad (N\Phi) = \mu_0 \dfrac{N^2}{l} iS$

Autoinducción: $\boxed{L = \dfrac{(N\Phi)}{i} = \mu_0 \dfrac{N^2 S}{l}} \quad \rightarrow \quad L = \mu_0 n^2 lS = \mu_0 n^2 \mathcal{V}$

41.2.b. Cable coaxial



Teorema de Ampère: $Bl = 2\pi r B = \mu_0 i \quad \rightarrow \quad B = \dfrac{\mu_0}{2\pi}\dfrac{i}{r}$

Flujo ligado: $\Phi = \displaystyle\int_S \boldsymbol{B}\bullet \mathrm{d}\boldsymbol{S} = \int_S B\,\mathrm{d}S =$

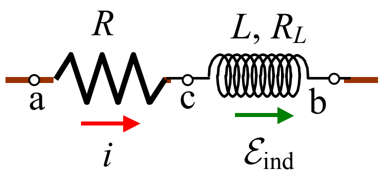
$$\frac{\mu_0 i}{2\pi}\int_{R_a}^{R_b} \frac{l\,\mathrm{d}r}{r} = \frac{\mu_0}{2\pi} il \int_{R_a}^{R_b}\frac{\mathrm{d}r}{r} = \frac{\mu_0}{2\pi} il \ln\frac{R_b}{R_a}$$

Autoinducción: $L = \dfrac{(\Phi)}{i} = \dfrac{\mu_0}{2\pi} l \ln\dfrac{R_b}{R_a} \quad \rightarrow \quad \boxed{\dfrac{L}{l} = \dfrac{\mu_0}{2\pi} l \ln\dfrac{R_b}{R_a}}$

§41.3.- Cálculo de la d.d.p

$$\begin{cases} v_{\underrightarrow{ac}} = iR - \left(\cancel{\mathcal{E}}\right) = iR \\ v_{\underrightarrow{cb}} = iR_L - \mathcal{E}_{\text{ind}} = \cancel{iR_L} + L\dfrac{\mathrm{d}i}{\mathrm{d}t} \end{cases} \quad \therefore \quad \boxed{v_L = L\frac{\mathrm{d}i}{\mathrm{d}t}} \quad \rightarrow \quad \begin{cases} \dfrac{\mathrm{d}i}{\mathrm{d}t} > 0 & (i\uparrow) & v_{cb} > 0 & v_c > v_b \\ \dfrac{\mathrm{d}i}{\mathrm{d}t} < 0 & (i\downarrow) & v_{cb} < 0 & v_c < v_b \end{cases}$$

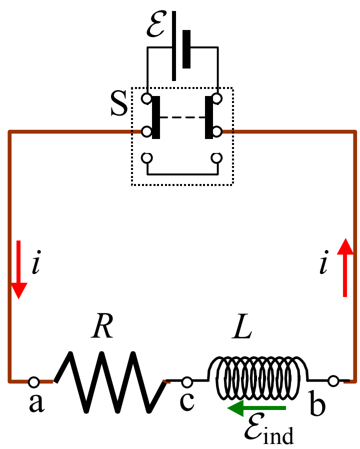
En general: $\boxed{v_{ab} = \sum iR - \sum(\mathcal{E} + \mathcal{E}_{\text{ind}}) \qquad \text{con} \quad \mathcal{E}_{\text{ind}} = -L\dfrac{\mathrm{d}i}{\mathrm{d}t}}$



§41.4.- Transitorios de cierre y de apertura en el circuito RL

41.4.a. Transitorios de cierre

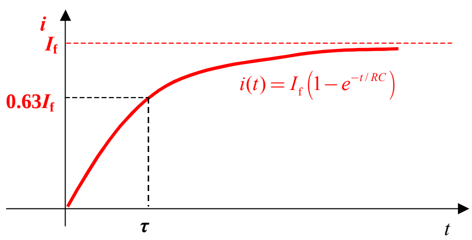
En el instante inicial el conmutador (S) está en la posición inferior y la autoinducción está "descargada". Al conectar el generador ($t = 0$), comienza a circular la corriente.



$$V_{ab} = v_R + v_L = Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} = \mathcal{E}$$

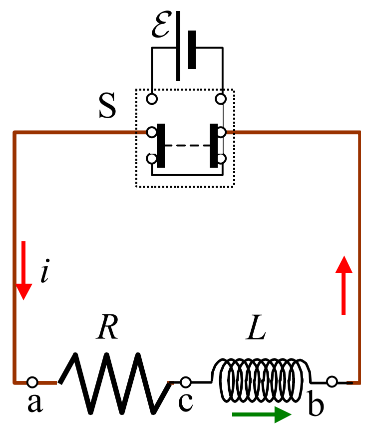
$$L\frac{\mathrm{d}i}{\mathrm{d}t} = \mathcal{E} - iR \quad \rightarrow \quad \frac{\mathrm{d}i}{\mathcal{E} - iR} = \frac{\mathrm{d}t}{L} \quad \rightarrow \quad \int_0^i \frac{\mathrm{d}i}{(\mathcal{E}/R) - i} = \frac{R}{L}\int_0^t \mathrm{d}t$$

$$\begin{cases} \boxed{i = \dfrac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right)} \\ \tau = \dfrac{L}{R} \end{cases} \quad \rightarrow \quad \begin{cases} t = 0^+ & i = 0 \\ t = \tau & i = I_0\left(1 - e^{-1}\right) = 0.63 I_f \\ t = \infty & i = I_f = \dfrac{\mathcal{E}}{R} \end{cases}$$



Constante de tiempo (τ): tiempo que transcurre en el proceso de cierre hasta que la corriente alcanza el 63% de su valor final.

41.4.b. Transitorio de apertura

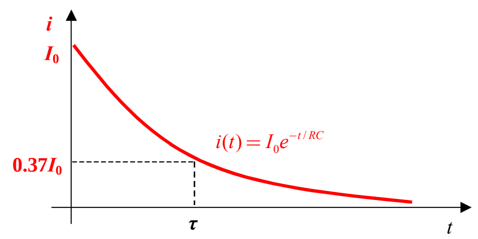


Por la autoinducción circula inicialmente una intensidad de corriente I_0. En el instante $t = 0$ se pasa el conmutador a la posición inferior, anulando la d.d.p. entre los puntos **a** y **b**.

$$V_{ab} = v_R + v_L = Ri + L\frac{di}{dt} = 0$$

$$\frac{di}{i} = -\frac{R}{L}dt \quad \rightarrow \quad \int_{I_0}^{i}\frac{di}{i} = -\frac{R}{L}\int_0^t dt \quad \rightarrow \quad \ln\frac{i}{I_0} = -\frac{R}{L}t$$

$$\left\{\begin{array}{l} \boxed{i = I_0 e^{-t/\tau}} \\ \boxed{\tau = L/R} \end{array}\right. \qquad \left\{\begin{array}{ll} t = 0^+ & i = I_0 \\ t = \tau & i = I_0/e = 0.37\, I_0 \\ t = \infty & i = 0 \end{array}\right.$$

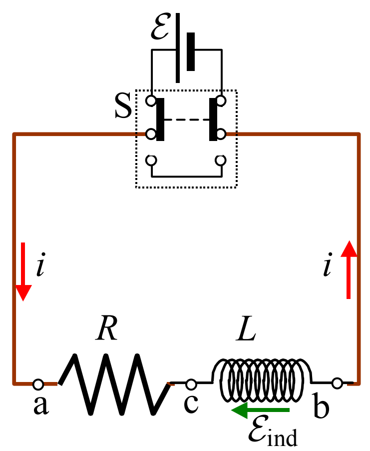


Constante de tiempo (τ): tiempo que transcurre en el proceso de apertura hasta que la corriente se reduce al 37% de su valor inicial.

§41.5.- Energía almacenada en una autoinducción

$$\mathcal{E} + \mathcal{E}_{\text{ind}} = Ri \quad \rightarrow \quad \mathcal{E} - L\frac{di}{dt} = Ri \quad \rightarrow \quad \mathcal{E} = Ri + L\frac{di}{dt}$$

$$P = i\mathcal{E} = i^2 R + Li\frac{di}{dt}$$

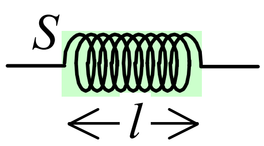


$Li\dfrac{di}{dt}$ es la energía almacenada por unidad de tiempo en la autoinducción (en el campo magnético).

$$\frac{dU_B}{dt} = Li\frac{di}{dt} \quad \rightarrow \quad dU_B = Li\, di \quad \rightarrow \quad \boxed{U_B = \int_0^i Li\, di = \frac{1}{2}Li^2}$$

§41.6.- Energía del campo magnético

$$u_B = \frac{U_B}{Sl} = \frac{1}{2}\frac{Li^2}{Sl} = \frac{1}{2}\frac{(\mu_0 n^2 lS)}{Sl}i^2 = \frac{1}{2}\mu_0 n^2 i^2 = \frac{1}{2}\frac{B^2}{\mu_0}$$



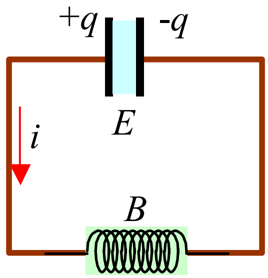
Densidad de energía en el campo magnético: $\boxed{u_B = \dfrac{1}{2}\dfrac{B^2}{\mu_0}}$

Ejemplo:

$$\begin{cases} E = 10^5\ \dfrac{\text{V}}{\text{m}} \\ B = 1\ \text{T} \end{cases} \rightarrow \quad \frac{u_B}{u_E} = \frac{B^2/\mu_0}{\varepsilon_0 E^2} = \frac{1}{\varepsilon_0\mu_0}\left(\frac{B}{E}\right)^2 = c^2\left(\frac{B}{E}\right)^2 = \left(c\frac{B}{E}\right)^2 = \left(\frac{3\times 10^8}{10^5}\right)^2 = 9\times 10^6$$

$$\therefore \quad u_B \gg u_E$$

§41.7.- Circuito oscilante



$$\frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{Q^2}{C} \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \quad \rightarrow \quad Li\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}q\dot{q} = 0$$

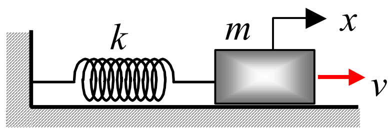
$$\therefore \quad L\dot{q}\ddot{q} + \frac{1}{C}q\dot{q} = 0 \quad \rightarrow \quad \left(\ddot{q} + \frac{1}{LC}q\right)\dot{q} = 0 \quad \rightarrow \quad \boxed{\ddot{q} + \frac{1}{LC}q = 0}$$

ec. Diferencial de las oscilaciones armónicas de la carga.

$$\begin{cases} q = Q\cos\omega t \\ i = -\omega Q\,\text{sen}\,\omega t = -I_{\text{máx}}\,\text{sen}\,\omega t \end{cases} \quad \text{con} \quad \begin{cases} \omega = \sqrt{\dfrac{1}{LC}} \\ I_{\text{máx}} = \omega Q \end{cases}$$

Analogías con el sistema masa-muelle:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \begin{cases} m \rightarrow L & k \rightarrow 1/C \\ v \rightarrow i & x \rightarrow q \end{cases}$$



§41.8.- Inductancia mutua

41.8.a. Definiciones

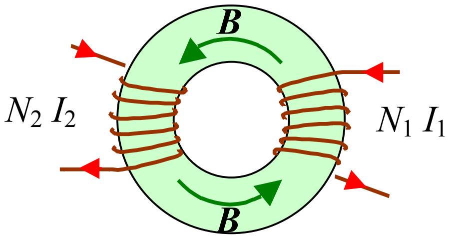
$$\Phi_i = \sum_{j=1}^{N}\Phi_{ij} \quad \rightarrow \quad \text{(Faraday)} \quad \rightarrow \quad \mathcal{E}_i = -\frac{\mathrm{d}\Phi_i}{\mathrm{d}t} = -\sum_{j=1}^{N}\frac{\mathrm{d}\Phi_{ij}}{\mathrm{d}t}$$

Circuitos en posiciones fijas:

$$\frac{\mathrm{d}\Phi_{ij}}{\mathrm{d}t} = \frac{\mathrm{d}\Phi_{ij}}{\mathrm{d}I_j}\frac{\mathrm{d}I_j}{\mathrm{d}t} = M_{ij}\frac{\mathrm{d}I_j}{\mathrm{d}t} \quad \rightarrow \quad \begin{cases} M_{ij} = \dfrac{\mathrm{d}\Phi_{ij}}{\mathrm{d}I_j} & \text{(inductamcia mutua)} \\ M_{ii} = L_i = \dfrac{\mathrm{d}\Phi_{ii}}{\mathrm{d}I_i} & \text{(autoinductancia)} \end{cases}$$

En definitiva: $\boxed{\mathcal{E}_i = -\sum_{j=1}^{N} M_{ij}\frac{\mathrm{d}I_j}{\mathrm{d}t}} \quad \rightarrow \quad \mathcal{E}_i = -L_i\frac{\mathrm{d}i}{\mathrm{d}t}$

41.8.b. Ejemplo: Doble enrollamiento toroidal



$$\begin{cases} B = \mu_0 \dfrac{N_1}{l} I_1 & \text{con } I_2 = 0 \\[2ex] B = \mu_0 \dfrac{N_2}{l} I_2 & \text{con } I_1 = 0 \end{cases}$$

$$\begin{cases} \Phi_{11} = N_1 B S = \mu_0 \dfrac{N_1^2 S}{l} I_1 \\[2ex] \Phi_{22} = N_2 B S = \mu_0 \dfrac{N_2^2 S}{l} I_2 \end{cases} \rightarrow \begin{cases} L_1 = \dfrac{\mathrm{d}\Phi_{11}}{\mathrm{d}I_1} = \mu_0 \dfrac{N_1^2 S}{l} \\[2ex] L_2 = \dfrac{\mathrm{d}\Phi_{22}}{\mathrm{d}I_2} = \mu_0 \dfrac{N_2^2 S}{l} \end{cases}$$

$$\begin{cases} \Phi_{21} = N_2 B S = \mu_0 \dfrac{N_1 N_2 S}{l} I_1 \\[2ex] \Phi_{12} = N_1 B S = \mu_0 \dfrac{N_1 N_2 S}{l} I_2 \end{cases} \rightarrow \begin{cases} M_{21} = \dfrac{\mathrm{d}\Phi_{21}}{\mathrm{d}I_1} = \mu_0 \dfrac{N_1 N_2 S}{l} \\[2ex] M_{21} = \dfrac{\mathrm{d}\Phi_{21}}{\mathrm{d}I_1} = \mu_0 \dfrac{N_1 N_2 S}{l} \end{cases} \rightarrow \quad M_{12} = M_{21} \quad \text{(siempre)}$$

$$M_{12}^2 = \left(\mu_0 \frac{N_1 N_2 S}{l} \right)^2 = \mu_0 \frac{N_1^2 S}{l} \mu_0 \frac{N_2^2 S}{l} = L_1 L_2 \quad \rightarrow \quad \boxed{M_{12} = \sqrt{L_1 L_2}}$$

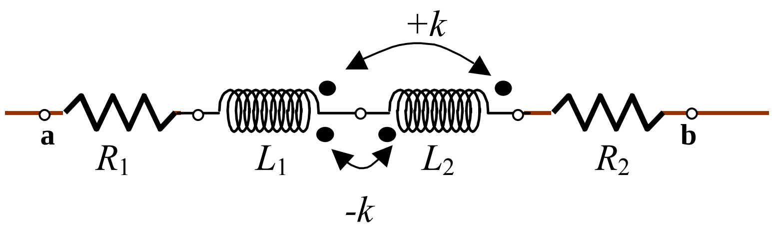
41.8.c. Fórmula de Newman

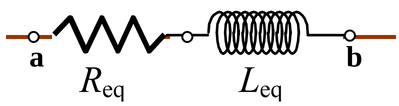
En general: $\boxed{M_{12} = k\sqrt{L_1 L_2}}$ (fórmula de Newman)

Factor de acoplamiento: $0 \leq k \leq 1$

§41.9.- Inductancias en serie y en paralelo

41.9.a. En serie



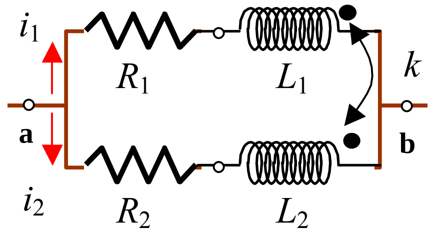


$$v_{\underline{ab}} = \sum iR - \sum \mathcal{E} = R_1 i + L_1 \frac{\mathrm{d}i}{\mathrm{d}t} \pm M \frac{\mathrm{d}i}{\mathrm{d}t} + R_2 i + L_2 \frac{\mathrm{d}i}{\mathrm{d}t} \pm M \frac{\mathrm{d}i}{\mathrm{d}t} =$$

$$= (R_1 + R_2) i + (L_1 + L_2 \pm 2M) \frac{\mathrm{d}i}{\mathrm{d}t} \qquad \therefore \qquad \boxed{R_{eq} = R_1 + R_2} \qquad \boxed{L_{eq} = L_1 + L_2 \pm 2M}$$

41.9.b. En paralelo

$$\begin{cases} v_{\underline{ab}} = R_1 i_1 + L_1 \dfrac{\mathrm{d}i_1}{\mathrm{d}t} \pm M \dfrac{\mathrm{d}i_2}{\mathrm{d}t} \\ v_{\underline{ab}} = R_2 i_2 + L_2 \dfrac{\mathrm{d}i_2}{\mathrm{d}t} \pm M \dfrac{\mathrm{d}i_1}{\mathrm{d}t} \end{cases} \quad \text{(ec. dif. acopladas)}$$



Caso simple: $R_1 = R_2 = 0$ (L puras)

$$\begin{cases} v = L_1 \dfrac{\mathrm{d}i_1}{\mathrm{d}t} \pm M \dfrac{\mathrm{d}i_2}{\mathrm{d}t} \quad \rightarrow \quad \dfrac{\mathrm{d}i_1}{\mathrm{d}t} = \dfrac{v}{L_1} \mp \dfrac{M}{L_1} \dfrac{\mathrm{d}i_2}{\mathrm{d}t} \\ v = L_2 \dfrac{\mathrm{d}i_2}{\mathrm{d}t} \pm M \dfrac{\mathrm{d}i_1}{\mathrm{d}t} \quad \rightarrow\rightarrow \quad v = L_2 \dfrac{\mathrm{d}i_2}{\mathrm{d}t} \pm \dfrac{Mv}{L_1} \mp \dfrac{M^2}{L_1} \dfrac{\mathrm{d}i_2}{\mathrm{d}t} \quad \rightarrow \quad L_1 v = L_1 L_2 \dfrac{\mathrm{d}i_2}{\mathrm{d}t} \pm Mv \mp M^2 \dfrac{\mathrm{d}i_2}{\mathrm{d}t} \end{cases}$$

$$\begin{cases} (L_1 \mp M) v = (L_1 L_2 \mp M^2) \dfrac{\mathrm{d}i_2}{\mathrm{d}t} \\ (L_2 \mp M) v = (L_1 L_2 \mp M^2) \dfrac{\mathrm{d}i_1}{\mathrm{d}t} \end{cases} \rightarrow \quad i = i_1 + i_2 \quad \rightarrow \quad (L_1 + L_{12} \mp 2M) v = (L_1 L_2 \mp M^2) \frac{\mathrm{d}i}{\mathrm{d}t}$$

$$\therefore \; v = \frac{L_1 L_2 \mp M^2}{L_1 + L_{12} \mp 2M} \left(\frac{\mathrm{d}i}{\mathrm{d}t} \right) \quad \rightarrow \quad \boxed{L_{\mathrm{eq}} = \frac{L_1 L_2 \mp M^2}{L_1 + L_{12} \mp 2M}}$$

$$M = k\sqrt{L_1 L_2} \rightarrow \begin{cases} k = 0 \rightarrow M = 0 \rightarrow L_{\mathrm{eq}} = \dfrac{L_1 L_2}{L_1 + L_2} \\ k = 1 \rightarrow M^2 = L_1 L_2 \rightarrow L_{\mathrm{eq}} = \dfrac{L_1 L_2 \mp L_1 L_2}{L_1 + L_2 \mp 2\sqrt{L_1 L_2}} \begin{cases} = 0 \\ = \dfrac{2 L_1 L_2}{L_1 + L_2 + 2\sqrt{L_1 L_2}} \end{cases} \end{cases}$$