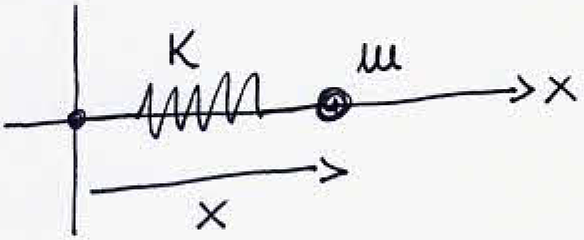
1 - DYNAMICAL SYSTEMS. OVERVIEW

1.1. CONTINUOUS SYSTEMS

$$\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{\mu}) \qquad \vec{x} = (x_1, x_2 \dots x_n)$$
$$\vec{\mu} = (\mu_1, \mu_2 \dots \mu_m)$$

Ex. 1: (MECHANICS). THE HARMONIC OSCILLATOR



$$\begin{cases} m\ddot{x} = -Kx \\ x(0) = A \\ \dot{x}(0) = 0 \end{cases}$$

THE ASSOCIATED DYNAMICAL SYSTEM IS OBTAINED BY DEFINING TWO NEW VARIABLES

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}\right\} \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{K}{m} x_1 = -\omega^2 x_1, \quad \omega = \sqrt{\frac{K}{m}} \end{aligned}$$

THE SOLUTIÓN IS

$$x_1(t) = A\cos(\omega t) \quad \leftarrow \text{POSITION}$$
$$x_2(t) = -A\omega \sin(\omega t) \quad \leftarrow \text{VELOCITY}$$

EXERCICES

a) PLOT x_1 vs t FOR $K = 10u$, $m = 100u$, $t \in [0, 10]$ (20 points), $A = 1u$

b) PLOT x_1 vs x_2 FOR $A \in [1, 10]$ (FIVE VALUES)

c) PLOT x_1 vs t WITH $K \in [1, 10]$ (FIVE POINTS)

Ex. 2 (MECHANICS). THE DAMPED HARMONIC OSCILLATOR

$$\begin{cases} m\ddot{x} = -kx - r\dot{x} \\ x(0) = A \\ \dot{x}(0) = 0 \end{cases} \rightarrow \left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \boxed{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{r}{m}x_2 \end{array}}$$

THE SOLUTION DEPENDS ON THE ROOTS OF THE DISCRIMINANT EQUATION

$$m\lambda^2 + r\lambda + k = 0 \rightarrow \lambda = \frac{-r \pm \sqrt{r^2 - 4km}}{2m} \begin{array}{l} \lambda_1 \\ \lambda_2 \end{array}$$

i) $r^2 - 4km > 0 \Rightarrow$ OVERDAMPED OSCILLATOR (NO OSC.)
ii) $r^2 - 4km = 0 \Rightarrow$ CRITICAL DAMPING (NO OSC.)
iii) $r^2 - 4km < 0 \Rightarrow$ UNDERDAMPED (OSCILLATIONS)

i) TWO REAL NEGATIVE ROOTS λ_1, λ_2, THEREFORE, THE SOLUTION IS,

$$\left.\begin{array}{l} x(t) = P_1 e^{\lambda_1 t} + P_2 e^{\lambda_2 t} \\ x(0) = P_1 + P_2 = A \\ \dot{x}(0) = \lambda_1 P_1 + \lambda_2 P_2 = 0 \end{array}\right\} x(t) = \frac{A}{1 - \frac{\lambda_1}{\lambda_2}} \cdot \left(e^{\lambda_1 t} - \frac{\lambda_1}{\lambda_2} e^{\lambda_2 t}\right)$$

ii) A SINGLE NEGATIVE ROOT $\lambda = -r/2m$. THE SOLUTION IS,

$$\left.\begin{array}{l} x(t) = e^{\lambda t} \cdot (P_1 t + P_2) \\ x(0) = P_2 = A \\ \dot{x}(0) = \lambda P_2 + P_1 = 0 \end{array}\right\} x(t) = e^{\lambda t}(A - A\lambda t) = A e^{\lambda t}(1 - \lambda t)$$

iii) TWO COMPLEX CONJUGATE ROOTS, THUS, THE SOLUTION IS

$$\lambda_{1,2} = -\frac{r}{2m} \pm j\sqrt{\frac{k}{m} - \frac{r^2}{4m^2}} = -\frac{r}{2m} \pm j\Omega$$

TAKING ADVANTAGE OF SOLUTION (i)

$$x(t) = \frac{A}{\lambda_2 - \lambda_1}\left(\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}\right) =$$

$$= \frac{A e^{-\frac{r}{2m}t}}{-2j\Omega}\left(\underbrace{\lambda_1^* e^{j\Omega t} - \lambda_1 e^{-j\Omega t}}_{z - z^* = 2j\, Im(z)}\right) =$$

$$= \frac{A e^{-\frac{r}{2m}t}}{-\cancel{2j}\Omega}\, \cancel{2j}\, Im\left(\lambda_1^* e^{j\Omega t}\right) =$$

$$= \frac{A e^{-\frac{r}{2m}t}}{-\Omega}\, \underbrace{Im\left[\left(-\frac{r}{2m} - j\Omega\right)\cdot\left(\cos\Omega t + j\sin\Omega t\right)\right]}_{-\frac{r}{2m}\sin(\Omega t) - \Omega\cos(\Omega t)} =$$

$$= A e^{-\frac{r}{2m}t}\left(\cos(\Omega t) + \frac{r}{2m\Omega}\sin(\Omega t)\right)$$

EXERCICES : LET $\omega_0^2 = \frac{K}{m}$, $\omega_f = \frac{r}{m}$, $A = 1$

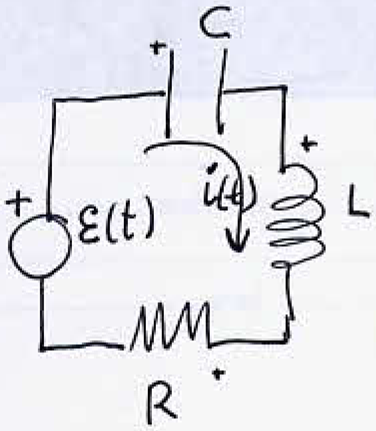
a) LET CONSIDER $\omega_f = 2$ AND $\omega_0 = 1$, (CASE (i))
PLOT X vs t WHEN $t \in [0, 10]$ (20 points)...

b) LET'S SUPPOSE NOW, $\omega_f = \omega_0 = 2$, (CASE (ii))
PLOT X vs t WHEN $t \in [0, 10]$ (20 points)

c) NOW, SUPPOSE THAT $\omega_f = 1$ AND $\omega_0 = 2$ (CASE (iii))
PLOT X vs t IN THE SAME RANGE THAT BEFORE.

d) IN THE CASE c), PLOT ALSO X_1 vs X_2 (CHANGE A IF NECESSARY)

EX. 3 (ELECTRIC) THE R-L-C SERIES CIRCUIT



$$\begin{cases} \frac{1}{C}\int_0^t i(t)\,dt + \frac{q(0)}{C} + L\frac{di}{dt} + i\cdot R - \mathcal{E}(t) = 0 \\ i(0) = 0 \end{cases}$$

THIS PROBLEM IS EQUIVALENT TO

$$\begin{cases} L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{d\mathcal{E}(t)}{dt} \\ i(0) = 0 \\ \frac{di}{dt}(0) = \frac{\mathcal{E}(0)}{L} - \frac{q(0)}{LC} \end{cases}$$

FOR THE SAKE OF SIMPLICITY LET'S SUPPOSE $\mathcal{E}(t) = \mathcal{E}$ AND $q(0) = 0$. THEN,

$$\begin{cases} L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0 \\ i(0) = 0 \\ \frac{di}{dt}(0) = \frac{\mathcal{E}}{L} \end{cases}$$

SIMILAR TO THE EQUATION OF THE PREVIOUS PROBLEM Ex. 2
THE DISCRIMINANT EQUATION IS NOW,

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0 \Rightarrow \lambda_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

THE DISSIPATIVE TERM. IS, IN THIS CASE, $R/2L$

LET'S CONSIDER THE "OVERDAMPED" SYSTEM.

$$\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} > 0 \;,\; \lambda_1, \lambda_2 < 0 \quad |\lambda_1| < |\lambda_2|$$

THE SOLUTION IS

$$\begin{cases} i(t) = P_1 e^{\lambda_1 t} + P_2 e^{\lambda_2 t} \\ i(0) = 0 = P_1 + P_2 \\ \frac{di}{dt}(0) = \frac{\mathcal{E}}{L} = \lambda_1 P_1 + \lambda_2 P_2 \end{cases}$$

$$\boxed{i(t) = \frac{\mathcal{E}/L}{\lambda_1 - \lambda_2}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)}$$

* **EXERCICE**: $R/2L = 2 \quad 1/LC = 1$ AND $\mathcal{E}/L = 1$

PLOT $i(t)$ vs t FOR $t \in [0, 10]$ (20 points at least)

LET'S NOW CONSIDER OSCILLATIONS $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0$

$$\lambda_{1,2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = \alpha \pm j\beta$$

$$i(t) = \frac{\mathcal{E}/L}{-2j\beta} e^{\alpha t} \underbrace{\left(e^{j\beta} - e^{-j\beta}\right)}_{2j\sin(\beta t)} = \frac{\mathcal{E}/L}{-\beta} e^{\alpha t} \sin(\beta t)$$

* **EXERCICE** $R/2L = 1$, $1/LC = 2$ AND $\mathcal{E}/L = 1$

PLOT i vs t FOR $t \in [0, 10]$ (20 points - or more)

CONVERTING O.D.E TO DYNAMICAL SYSTEM FORM

1.- NON-AUTONOMOUS TO AUTONOMOUS

LET' CONSIDER THE O.D.E

$$A\frac{d^2x}{dt^2} + B\frac{dx}{dt} + C\cdot x = f(t)$$

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= t \end{aligned}\right\} \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (f(x_3) - Cx_1 - Bx_2)/A \\ \dot{x}_3 &= 1 \end{aligned}$$

2- AN O.D.E SYSTEM TO DYNAMICAL SYSTEM FORM

LET'S CONSIDER THE SYSTEM OF ODE.

$$\begin{cases} A\dfrac{d^2x}{dt^2} + B\dfrac{d^2y}{dt^2} = f\left(\dfrac{dx}{dt}, \dfrac{dy}{dt}, x, y, t\right) \\[2ex] B\dfrac{d^2x}{dt^2} + D\dfrac{d^2y}{dt} = g\left(\dfrac{dx}{dt}, \dfrac{dy}{dt}, x, y, t\right) \end{cases}$$

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= y \\ x_4 &= \dot{y} \\ x_5 &= t \end{aligned}\right\} \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= F(x_1, x_2, x_3 x_4, x_5) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= G(x_1, x_2, x_3, x_4, x_5) \\ \dot{x}_5 &= 1 \end{aligned}$$

THE FUNCTIONS F AND G ARE THE SOLUTIONS OF THE LINEAR SYSTEM

$$\begin{cases} A\dot{x}_2 + B\dot{x}_4 = f(x_1, x_2, x_3, x_4, x_5) \\ C\dot{x}_2 + D\dot{x}_4 = g(x_1, x_2, x_3, x_4, x_5) \end{cases}$$

WHAT ABOUT THE INITIAL CONDITIONS?

SUPPOSE THAT ONE HAVE INITIALLY

$$x(0) = x_0 \qquad \dot{x}(0) = v_{0x}$$
$$y(0) = y_0 \qquad \dot{y}(0) = v_{0y}$$

THEN, THE INITIAL CONDITIONS FOR THE DYNAMICAL SYSTEM RESULT:

$$\begin{cases} x_1(0) = x_0 \\ x_2(0) = v_{0x} \\ x_3(0) = y_0 \\ x_4(0) = v_{0y} \end{cases}$$

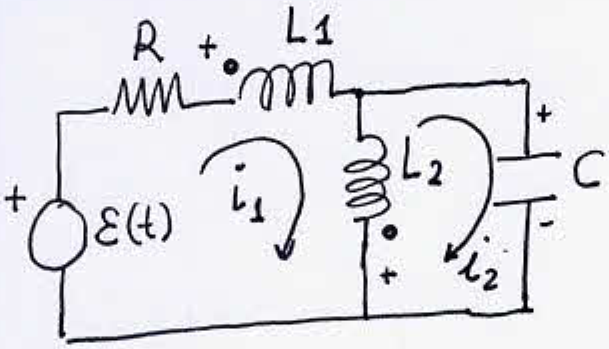
EXERCICES

1) CONVERT $L\dfrac{d^2 i}{dt^2} + R\dfrac{di}{dt} + \dfrac{i}{C} = \mathcal{E}_0 w \cos(wt)$

2) CONVERT

$$\begin{cases} \dfrac{d^2x}{dt^2} + \dfrac{d^2y}{dt^2} + \dfrac{dx}{dt} + \dfrac{dy}{dt} = \alpha \cdot t \\ \dfrac{d^2x}{dt^2} - \dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} = 0 \end{cases}$$

Ex. 4 (ELECTRIC) ELECTRIC TRANSIENTS (WITH TRANSF...



$i_1(0)=0$

$i_2(0)=0$

$$i_1 R + \underbrace{L_1\frac{di_1}{dt} + M\frac{d(i_2-i_1)}{dt}}_{\mathcal{E}_{L_1}(t)} - \underbrace{L_2\frac{d(i_1-i_2)}{dt} + M\frac{di_1}{dt}}_{\mathcal{E}_{L_2}(t)} = \mathcal{E}(t)$$

$$\underbrace{L_2\frac{d(i_2-i_1)}{dt} + M\frac{di_1}{dt}}_{\mathcal{E}_{L_2}(t)} + \frac{1}{C}\int_0^t i_2\,dt \pm \frac{q(0)}{C} = 0$$

DERIVING ONCE WITH RESPECT TO TIME, ONE CAN OBTAIN

$$\boxed{\left.\begin{aligned} (L_1 - L_2)\frac{d^2 i_1}{dt^2} + (M+L_2)\frac{d^2 i_2}{dt^2} + R\frac{di_1}{dt} &= \frac{d\mathcal{E}}{dt} \\ L_2\frac{d^2 i_2}{dt^2} + (M-L_2)\frac{d^2 i_1}{dt^2} + \frac{i_2}{C} &= 0 \end{aligned}\right]\ (D.S)}$$

TOGETHER WITH THE INITIAL CONDITIONS

$i_1(0)=0 \quad i_2(0)=0$

$$\left.\begin{aligned} (L_1 - L_2)\frac{di_1(0)}{dt} + (M+L_2)\frac{di_2(0)}{dt} &= \mathcal{E}(0) \\ (M-L_2)\frac{di_1(0)}{dt} + L_2\frac{di_2(0)}{dt} &= 0 \end{aligned}\right\}\ (C.I)$$

THE DYNAMICAL SYSTEM FROM THE PREVIOUS EQUATIONS WOULD HAVE FOUR STATE VARIABLES AND MORE THAN FIVE PARAMETERS

$$\vec{x} = (x_1, x_2, x_3, x_4) \qquad \vec{\mu} = (R, L_1, L_2, C, \underset{\substack{\uparrow \\ 2 \text{ OR MORE}}}{\mathcal{E}(t)})$$

LET SUPPOSE THAT $\mathcal{E}(t) = \mathcal{E}_0 \cdot \cos(\omega t)$.
IF IT IS SO, THE SYSTEM WOULD BE NON-AUTONOMOUS BECAUSE OF THE EXPLICIT TIME DEPENDENCY IN $\mathcal{E}(t)$

$$\left.\begin{aligned} x_1 &= l_1 \\ x_2 &= \frac{dl_1}{dt} \\ x_3 &= l_2 \\ x_4 &= \frac{dl_2}{dt} \end{aligned}\right\} \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha x_2 + \beta x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \gamma x_2 + \delta x_3 \end{aligned}$$

WHERE α, β, γ, AND δ ARE THE SOLUTIONS OF (D.S) REFORMULATED AS

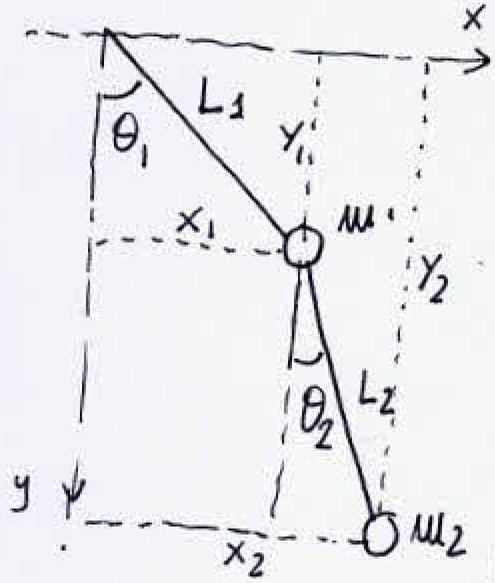
$$\boxed{\begin{aligned} (L_1 + L_2 - 2M)\,\dot{x}_2 + (M + L_2)\,\dot{x}_4 &= \frac{d\mathcal{E}}{dt} - R \cdot x_2 \\ (M - L_2)\,\dot{x}_2 + L_2\,\dot{x}_4 &= -\frac{1}{C} \cdot x_3 \end{aligned}}$$

AND THE INITIAL CONDITIONS BECOME IN

$$\begin{aligned} x_1(0) &= 0 \\ x_2(0) &= f(\mathcal{E}(0), L_1, L_2, C, R) \\ x_3(0) &= 0 \\ x_4(0) &= g(\mathcal{E}(0), L_1, L_2, C, R) \end{aligned}$$

OBTAINED AFTER SOLVING (C.I)

Ex. 5: THE DOUBLE PENDULUM



THE MOTION EQUATIONS WILL BE OBTAINED BY THE LAGRANGIAN FORMALISM.

$$\mathcal{L} = T - V = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - m_1 g y_1 - m_2 g y_2$$

$$\begin{cases} x_1 = L_1 \sin\theta_1 \Rightarrow \dot{x}_1 = L_1 \dot{\theta}_1 \cos\theta_1 \\ y_1 = L_1 \cos\theta_1 \Rightarrow \dot{y}_1 = -L_1 \dot{\theta}_1 \sin\theta_1 \\ x_2 = x_1 + L_2 \sin\theta_2 \Rightarrow \dot{x}_2 = \dot{x}_1 + L_2 \dot{\theta}_2 \cos\theta_2 \\ y_2 = y_1 + L_2 \cos\theta_2 \Rightarrow \dot{y}_2 = \dot{y}_1 - L_2 \dot{\theta}_2 \sin\theta_2 \end{cases}$$

IT IS STRAIGHTFORWARD TO SHOW THAT

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = L_1^2 \dot{\theta}_1^2$$

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = L_1^2 \dot{\theta}_1^2 + L_2^2 \dot{\theta}_2^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\mathcal{L} = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(L_1^2 \dot{\theta}_1^2 + L_2^2 \dot{\theta}_2^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right)$$
$$+ m_1 g L_1 \cos\theta_1 + m_2 g \left(L_1 \cos\theta_1 + L_2 \cos\theta_2\right)$$

AND THE MOTION EQUATIONS ARE OBTAINED BY

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0 \quad \text{AND} \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = 0$$

$$(m_1 + m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)\right] +$$
$$+ m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g L_1 \sin\theta_1 = 0$$

$$m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \left[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 \cdot (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)\right] -$$
$$- m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g L_2 \sin\theta_2 = 0$$

SIMPLIFYING, RESULTS...

$$(m_1+m_2)L_1^2\ddot{\theta}_1 + m_2L_1L_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2L_1L_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) +$$
$$+(m_1+m_2)gL_1\sin\theta_1 = 0$$

$$m_2 l_2^2\ddot{\theta}_2 + m_2L_1L_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2L_1L_2\dot{\theta}_1^2\sin(\theta_1-\theta_2) +$$
$$+ m_2gL_2\sin\theta_2 = 0$$

WHICH CAN BE DIVIDED BY L_1 AND L_2 RESPECTIVELY, OBTAINING THE FINAL FORM.

$$\boxed{\begin{aligned}&(m_1+m_2)L_1\ddot{\theta}_1 + m_2L_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2L_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + (m_1+m_2)g\sin\theta_1 = 0\\ &m_2L_2\ddot{\theta}_2 + m_2L_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2L_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) + m_2g\sin\theta_2 = 0\end{aligned}}$$

TOGETHER WITH $\theta_1(0)=\theta_{10}$, $\theta_2(0)=\theta_{20}$ $\dot{\theta}_1(0)=\dot{\theta}_2(0)=0$

THE FINAL STEP IS CONVERT IT TO A DYNAMICAL SYSTEM FORM.

$$\left.\begin{aligned}\theta_1 &= x_1\\ \dot{\theta}_1 &= x_2\\ \theta_2 &= x_3\\ \dot{\theta}_2 &= x_4\end{aligned}\right\} \quad \begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= f(x_1,x_2,x_3,x_4)\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= g(x_1,x_2,x_3,x_4)\end{aligned}$$

WHERE f AND g ARE OBTAINED BY SOLVING THE SYSTEM.

$$(m_1+m_2)L_1\dot{x}_2 + m_2L_2\dot{x}_4\cdot\cos(x_1-x_3) = -m_2L_2x_4^2\sin(x_1-x_3) - (m_1+m_2)g\cdot\sin(x_1)$$
$$m_2L_1\cos(x_1-x_3)\dot{x}_2 + m_2L_2\dot{x}_4 = m_2L_1x_2^2\sin(x_1-x_3) - m_2g\sin x_3$$

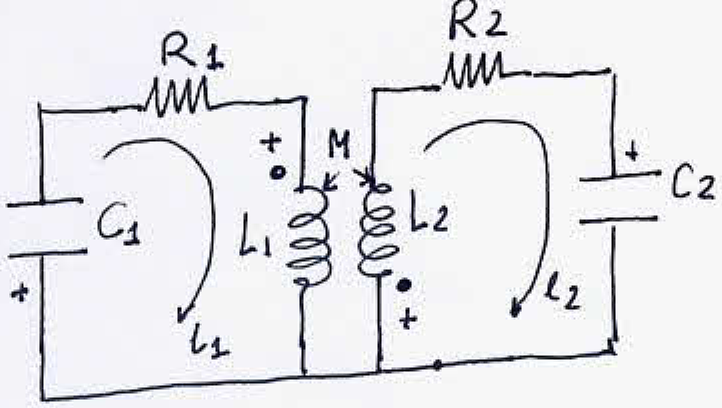
$$\dot{x}_2 = \frac{-m_2 L_2 x_4^2 \sin(x_1-x_3) - (m_1+m_2) g \sin x_1 - m_2 L_1 x_2^2 \sin(x_1-x_3)\cdot \cos(x_1-x_3) + m_2 g \sin x_3 \cdot \cos(x_1-x_3)}{(m_1+m_2)L_1 - m_2 L_1 \cos^2(x_1-x_3)}$$

$$\dot{x}_4 = \frac{m_2 L_1 x_2^2 \sin(x_1-x_3) - m_2 g \sin x_3}{m_2 L_2} - \frac{m_2 L_1 \cos(x_1-x_3)}{m_2 L_2}\cdot \dot{x}_2$$

FROM THIS EXPRESSIONS, A NUMERICAL SCHEME USING MATLAB CAN BE WRITTEN TO SOLVE THE DOUBLE PENDULUM MOTION.

Ex.6 : OSCILLATIONS IN ELECTRICAL CIRCUITS

WE STUDY NOW THE OSCILLATIONS ARISING IN A ELECTRICAL CIRCUIT WHERE COUPLED INDUCTANCES ARE PRESENT.



$$L_1 \frac{di_1}{dt} + M\frac{di_2}{dt} + i_1 R_1 + \frac{1}{C_1}\int_0^t i_1 dt \pm \frac{q_1(0)}{C_1} = 0$$

$$L_2 \frac{di_2}{dt} + M\frac{di_1}{dt} + i_2 R_2 + \frac{1}{C_2}\int_0^t i_2 dt = 0$$

WHERE $q_1(0)$ IS THE INITIAL CHARGE OF C_1, AND $i_1(0) = i_2(0) = 0$

DERIVING WITH RESPECT TO TIME, LEADS TO

$$\boxed{\begin{aligned} &L_1 \frac{d^2 i_1}{dt^2} + M\frac{d^2 i_2}{dt^2} + R_1\frac{di_1}{dt} + \frac{1}{C_1} i_1 = 0 \\ &L_2 \frac{d^2 i_2}{dt^2} + M\frac{d^2 i_1}{dt^2} + R_2\frac{di_2}{dt} + \frac{1}{C_2} i_2 = 0 \end{aligned}}$$

THUS, THE ASSOCIATED DYNAMICAL SYSTEM WOULD HAVE THE FOLLOWING FORM

$$
\left.\begin{aligned}
x_1 &= \ell_1 \\
x_2 &= d\ell_1/dt \\
x_3 &= \ell_2 \\
x_4 &= d\ell_2/dt
\end{aligned}\right\}
\quad
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f(x_1, x_2, x_3, x_4) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= g(x_1, x_2, x_3, x_4)
\end{aligned}
$$

WHERE THE UNKNOWN FUNCTIONS f AND g ARE OBTAINED FROM THE SOLUTIONS OF THE LINEAR SYSTEM

$$
\begin{cases}
L_1 \dot{x}_2 + M \dot{x}_4 = -\frac{1}{C_1} x_1 - R_1 \cdot x_2 \\
M \dot{x}_2 + L_2 \dot{x}_4 = -\frac{1}{C_2} x_3 - R_2 x_4
\end{cases}
$$

$$
\dot{x}_2 = \frac{1}{L_1 L_2 - M^2} \cdot \left(-\frac{L_2}{C_1} x_1 - R_1 L_2 x_2 + \frac{M}{C_2} x_3 + R_2 M x_4 \right)
$$

$$
\dot{x}_4 = \frac{1}{L_1 L_2 - M^2} \cdot \left(-\frac{L_1}{C_2} x_3 - L_1 R_2 x_4 + \frac{M}{C_1} x_1 + R_1 M x_2 \right)
$$

TOGETHER WITH THE INITIAL CONDITIONS

$$
x_1(0) = 0 \ ; \ x_2(0) = A \ ; \ x_3(0) = 0 \ ; \ x_4(0) = B
$$

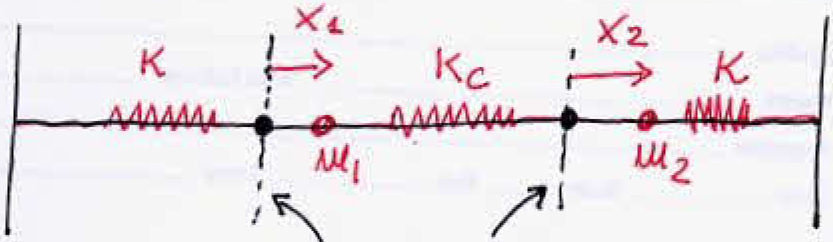
WHERE A AND B ARE THE SOLUTIONS OF

$$
\left.\begin{aligned}
L_1 x_2(0) + M x_4(0) &= \mp \frac{q_1(0)}{C_1} \\
M x_2(0) + L_2 x_4(0) &= 0
\end{aligned}\right\}
\quad
A = \frac{\mp q_1(0) L_2 / C_1}{L_1 L_2 - M^2}
$$

$$
B = \frac{\pm q_1(0) \cdot M / C_1}{L_1 L_2 - M^2}
$$

EX. 7 COUPLED MECHANICAL OSCILLATORS

LET'S CONSIDER THE FOLLOWING DIAGRAM



$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2} K x_1^2 + \frac{1}{2} K_c (x_1 - x_2)^2 + \frac{1}{2} K x_2^2$$

$$\mathcal{L} = T - V = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} K x_1^2 - \frac{1}{2} K_c (x_1 - x_2)^2 - \frac{1}{2} K x_2^2$$

AND THE EQUATIONS OF MOTION ARE

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1 \dot{x}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -K x_1 - K_c (x_1 - x_2) = -(K + K_c) x_1 + K_c x_2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_2 \dot{x}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = + K_c (x_1 - x_2) - K x_2 = -(K + K_c) x_2 + K_c x_1$$

THUS, THE EQUATIONS HAVE THE FORM.

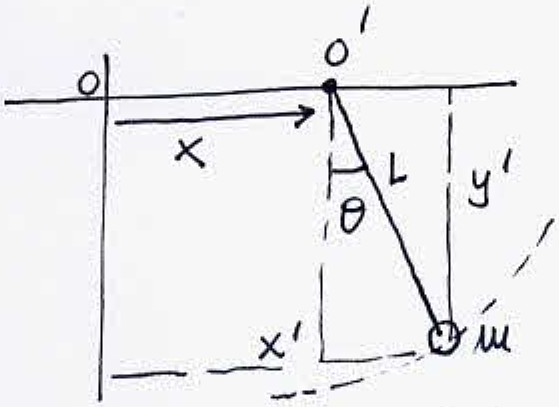
$$\boxed{\begin{aligned} m_1 \ddot{x}_1 + (K + K_c) x_1 - K_c x_2 = 0 \\ m_2 \ddot{x}_2 + (K + K_c) x_2 - K_c x_1 = 0 \end{aligned}}$$

THE DYNAMICAL SYSTEM ASSOCIATED IS

$$\left.\begin{aligned} y_1 &= x_1 \\ y_2 &= \dot{x}_1 \\ y_3 &= x_2 \\ y_4 &= \dot{x}_2 \end{aligned}\right\} \qquad \begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -\frac{(K+K_c)}{m_1} y_1 + \frac{K_c}{m_1} y_3 \\ \dot{y}_3 &= y_4 \\ \dot{y}_4 &= -\frac{(K+K_c)}{m_2} y_3 + \frac{K_c}{m_2} y_1 \end{aligned}$$

Ex. 8 FORCED PENDULUM

LET'S CONSIDER A PENDULUM WHOSE PIVOT IS SUBJECTED TO AN OSCILLATORY MOVEMENT AT A FIXED FREQUENCY.



$$x(t) = x_0 \cos \omega t$$

$$x' = x + L \sin\theta \Rightarrow \dot{x}' = \dot{x} + L\dot{\theta}\cos\theta$$

$$y' = L\cos\theta \Rightarrow \dot{y}' = -L\dot{\theta}\sin\theta$$

$$\mathcal{L} = \frac{1}{2} m (\dot{x}'^2 + \dot{y}'^2) + m g y' = \frac{1}{2} m (\dot{x}^2 + L^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}L\cos\theta) + m g L \cos\theta$$

AND THE MOTION EQUATION IS

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m L^2 \dot{\theta} + m \dot{x} L \cos\theta \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = m L^2 \ddot{\theta} + m \ddot{x} L \cos\theta - m \dot{x} L \dot{\theta} \sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = - m g L \sin\theta - m \dot{x} \dot{\theta} L \sin\theta$$

$$\longrightarrow \quad m L^2 \ddot{\theta} + m \ddot{x} L \cos\theta + m g L \sin\theta = 0$$

WHICH ONCE SIMPLIFIED, RESULTS

$$\boxed{\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{x_0 w^2}{L}\cos\theta \cdot \cos wt}$$

$$\theta(0) = \theta_0$$
$$\dot{\theta}(0) = 0.$$

IF A DAMPING TERM IS ADDED, THE RESULTING EQUATION WOULD BE

$$\boxed{\ddot{\theta} + \frac{r}{m}\dot{\theta} + \frac{g}{L}\sin\theta = \frac{x_0 w^2}{L}\cos\theta \cdot \cos wt}$$

AS IT IS SEEN, CORRESPONDS TO A NON-AUTONOMOUS DYNAMICAL SYSTEM. THE ASSOCIATED AUTONOMOUS ONE, IS BUILT AS,

$$\left.\begin{array}{l} x_1 = \theta \\ x_2 = \dot{\theta} \\ x_3 = t \end{array}\right\} \quad \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{x_0 w^2}{L}\cos x_1 \cdot \cos(w \cdot x_3) - \overset{G}{\dfrac{r}{m}} x_2 - \dfrac{g}{L}\sin x_1 \\ \dot{x}_3 = 1 \end{array}$$

$$x_1(0) = \theta_0, \quad x_2(0) = 0 \quad x_3(0) = 0$$

FOR ADEQUATE VALUE OF THE PARAMETERS, ONE CAN OBTAIN CHAOTIC BEHAVIOR.